February 2005
RM3-TH/05-2
hep-ph/0503107

Light-quark Loops in $K \to \pi\nu\bar{\nu}$

Gino Isidori,[1] Federico Mescia,[1,2] and Christopher Smith,[1]

[1] *INFN, Laboratori Nazionali di Frascati, I-00044 Frascati, Italy*

[2] *Dip. di Fisica, Università di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy*

Abstract

We present a comprehensive analysis of the contributions to $K \to \pi\nu\bar{\nu}$ decays not described by the leading dimension-six effective Hamiltonian. These include both dimension-eight four-fermion operators generated at the charm scale, and genuine long-distance contributions which can be described within the framework of chiral perturbation theory. We show that a consistent treatment of the latter contributions, which turn out to be the dominant effect, requires the introduction of new chiral operators already at $O(G_F^2 p^2)$. Using this new chiral Lagrangian, we analyze the long-distance structure of $K \to \pi\nu\bar{\nu}$ amplitudes at the one-loop level, and discuss the role of the dimension-eight operators in the matching between short- and long-distance components. From the numerical point of view, we find that these $O(G_F^2 \Lambda_{\mathrm{QCD}}^2)$ corrections enhance the SM prediction of $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$ by about $\approx 6\%$.

1 Introduction

Flavour-changing neutral-current (FCNC) processes and the related GIM mechanism [1] are one of the most fascinating aspects of flavour physics. Within the Standard Model (SM), FCNC amplitudes are strongly suppressed and often completely dominated by short-distance dynamics. In this case, their precise study allows to perform very stringent tests of the model and ensures a large sensitivity to possible new degrees of freedom [3, 4].

On general grounds, we can distinguish two types of FCNC transitions: those where the leading short-distance amplitude exhibits a power-like GIM mechanism, and those where the GIM suppression is only logarithmic. This distinction plays a key role in kaon physics, where long-distance effects are enhanced by the hierarchy of the Cabibbo-Kobayashi-Maskawa (CKM) matrix [2] and could easily spoil the short-distance structure of FCNC amplitudes. Only in the case of a power-like GIM mechanism, or a power-like suppression of light-quark contributions, long-distance effects can be kept under good theoretical control.

The quark-level transition $s \to d\nu\bar{\nu}$ is the prototype of FCNC amplitudes with a power-like GIM mechanism. The leading contributions to this amplitude are genuine one-loop electroweak effects, which are usually encoded in the following $O(G_F^2)$ effective Hamiltonian (see e.g. Ref. [4]):

$$\mathcal{H}_{eff}^{(6)} = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2\theta_W} \left[V_{ts}^* V_{td} X_t(x_t) + V_{cs}^* V_{cd} X_c^l(x_c) \right] (\bar{s}d)_{V-A} (\bar{\nu}_l \nu_l)_{V-A} \; , \tag{1}$$

with $x_q = m_q^2/M_W^2$ As a consequence of the power-like GIM mechanism, the coefficient functions of the unique dimension-6 operator in Eq. (1) behave as $X_q(x_q) \propto x_q$ (up to logarithmic and subleading power corrections).[1] This implies that the term proportional to $V_{ts}^* V_{td}$, which is enhanced by the large top-quark mass and can be precisely computed in perturbation theory at the electroweak scale [5, 6, 7], is the dominant contribution.

In the case of CP-conserving transitions, such as the $K^+ \to \pi^+\nu\bar{\nu}$ decay, the charm contribution in $\mathcal{H}_{eff}^{(6)}$ cannot be neglected. Indeed the power suppression of X_c^l with respect to X_t is partially compensated by the large CKM coefficient ($|V_{cs}^* V_{cd}| \approx 10^3 \times |V_{ts}^* V_{td}|$). Moreover, charm quarks remain dynamical degrees of freedom for a large range of energies below the electroweak scale. This imply an enhancement factor due to large logs and a stronger sensitivity to QCD corrections in X_c^l. Thanks to the NLO calculation of Ref. [5, 7], X_c^l is known with a relative precision of about 18%. Since the charm contribution amounts to about 30% of the total magnitude of $\mathcal{A}(s \to d\nu\bar{\nu})_{\rm SM}$, the NLO uncertainty translates into an error of about 10% in the SM estimate of $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$. This type of uncertainty can possibly be reduced to below 4% with a NNLO calculation of X_c^l [4].

Aiming to get a few % precision on the $K^+ \to \pi^+\nu\bar{\nu}$ amplitude, it becomes important to address the question of the subleading terms not described by the effective Hamiltonian in Eq. (1). In particular, $\mathcal{H}_{eff}^{(6)}$ does not allow to evaluate in a systematic way the

[1] This behavior illustrates the $O(G_F^2)$ structure of $\mathcal{H}_{eff}^{(6)}$: $G_F\alpha/(2\sqrt{2}\pi\sin^2\theta_W) \times x_q = G_F^2 m_q^2/(2\pi^2)$

contributions of $O(G_F^2\Lambda_{\rm QCD}^2)$ to the $K^+ \to \pi^+\nu\bar{\nu}$ amplitude. Naively, these contributions are parametrically suppressed only by $O(\Lambda_{\rm QCD}^2/m_c^2) \approx O(10\%)$ with respect to the charm contribution in Eq. (1). Therefore, they are not trivially negligible at the few % level of accuracy. The purpose of this paper is a systematic analysis of this type of effects.

The relevant subleading contributions to $K \to \pi\nu\bar{\nu}$ can be safely computed in the limit $V_{td} = 0$ (or in the limit where charm- and up-quark loops appear with the same CKM coefficient) and can be divided into two groups:

i. $O(1)$ (tree-level) matrix elements of local FCNC operators of dimension eight, such as $(\bar{s}\Gamma\partial s) \times (\bar{\nu}\Gamma\partial\nu)$, appearing in the $O(G_F^2)$ Hamiltonian;

ii. $O(G_F)$ (beyond tree-level) matrix elements of the $\Delta S = 1$ dimension-6 four-fermion operators appearing in the $O(G_F)$ effective Hamiltonian.

The distinction between these two types of effects depends on the choice of the renormalization scale for the effective four-fermion theory. For instance, choosing a renormalization scale well above the charm mass, one can essentially neglect the dimension-8 FCNC operators and encode all the effects via appropriate matrix-elements of the $\Delta S = 1$ effective Hamiltonian (where charm quarks are still treated as dynamical degrees of freedom). This approach would be the most natural choice in view of a calculation of these matrix elements by means of lattice QCD (probably the ultimate way to address this problem). Waiting for such a calculation on the lattice, here we adopt a different procedure and choose a renormalization scale for the effective four-fermion operators below the charm mass. As shown in Ref. [8, 9], this is the most natural choice in view of a fully analytic approach to the problem.

Concerning the construction of the dimension-8 four-fermion Hamiltonian, we completely confirm the results of Ref. [9]. However, we substantially extend this work by analysing the impact of the genuine long-distance component of the amplitude, namely the matrix-elements of $\Delta S = 1$ four-fermion operators (where only u, d and s quarks are treated as dynamical degrees of freedom). The latter component cannot be computed at the partonic level and the best analytic approach to evaluate its size is provided by chiral perturbation theory (CHPT). Several authors have already addressed the issue of long-distance effects in the $K^+ \to \pi^+\nu\bar{\nu}$ amplitude within the framework of CHPT [10, 11, 12, 13, 14]. However, as we shall show, all previous attempts to address this problem were not complete and, in particular, were not able to discuss the matching between short- and long-distance components of the amplitudes.

The paper is organized as follows: in section 2 we analyse the structure of the dimension-8 four-fermion Hamiltonian. The main new results are contained in section 3, where we construct the effective Lagrangian relevant to evaluate FCNCs of $O(G_F^2)$, we evaluate the long-distance components of $K \to \pi\nu\bar{\nu}$ amplitudes up to $O(G_F^2 p^4)$, and we discuss the matching between the chiral approach and the four-fermion operators. These results are used in section 4 to address the numerical impact on $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$. The conclusion are summarized in section 5.

2 The $O(G_F^2)$ four-fermion effective Hamiltonian

Since we are interested only in contributions generated by up- and charm-quark loops (namely we neglect the corrections of $O(\Lambda_{\rm QCD}/m_t^2)$), we can set $V_{td}=0$. In this limit, CKM unitarity allows to express all the relevant contributions in terms of one independent CKM combination: $\lambda_c = V_{cs}^* V_{cd} = -V_{us}^* V_{ud}$. As discussed in Ref. [5, 9], the central point for the construction of the low-energy effective theory is the expansion in terms of local operators of the following T-products,[2]

$$O_1^Z \;=\; -i\int d^4x\; T\left[Q_1^{cc}(x)\; Q_Z^{cc\nu\nu}(0)\; - Q_1^{uu}(x)\; Q_Z^{uu\,\nu\nu}(0)\right]\;, \tag{2}$$

$$O_2^Z \;=\; -i\int d^4x\; T\left[Q_2^{cc}(x)\; Q_Z^{cc\,\nu\nu}(0)\; - Q_2^{uu}(x)\; Q_Z^{uu\,\nu\nu}(0)\right]\;, \tag{3}$$

$$O_l^B \;=\; -i\int d^4x\; T\left[Q^{cl}(x)\; Q^{lc}(0)\; - Q^{ul}(x)\; Q^{lu}(0)\right]\;, \tag{4}$$

whose leading term is given by

$$Q_l^{(6)} = \bar{s}\gamma^\mu(1-\gamma_5)d\;\bar{\nu}_l\gamma_\mu(1-\gamma_5)\nu_l\;. \tag{5}$$

Here

$$\begin{aligned} Q_1^{qq} &= \bar{s}_i\gamma^\mu(1-\gamma_5)q_j\;\bar{q}_j\gamma_\mu(1-\gamma_5)d_i\;,\\ Q_2^{qq} &= \bar{s}_i\gamma^\mu(1-\gamma_5)q_i\;\bar{q}_j\gamma_\mu(1-\gamma_5)d_j\;, \end{aligned} \tag{6}$$

denote the leading $\Delta S=1$ four-quark operators ($q=u,c$),

$$Q_Z^{qq\nu\nu} = \bar{q}_k\gamma^\mu\left[(1-\gamma_5)-\frac{8}{3}\sin^2\theta_W\right]q_k\;\bar{\nu}_l\gamma_\mu(1-\gamma_5)\nu_l \tag{7}$$

is the effective neutral-current coupling induced by the integration of the Z boson, and

$$\begin{aligned} Q_3^{ql} &= \bar{s}\gamma^\mu(1-\gamma_5)q\;\bar{\nu}_l\gamma_\mu(1-\gamma_5)l\\ Q_4^{lq} &= \bar{l}\gamma^\mu(1-\gamma_5)\nu_l\;\bar{q}\gamma_\mu(1-\gamma_5)d \end{aligned} \tag{8}$$

are the effective charged-current couplings induced by integration of the $W^\pm$ bosons. Note that, even if we are interested in dimension-8 operators, we work at $O(G_F^2)$ and we can safely use a point-like propagator in the case of both Z and $W^\pm$ bosons. The T-products in Eqs. (2)–(4) correspond to the diagrams in Figure 1.

The first two steps necessary for the construction of the effective theory, namely the determination of the initial conditions at $\mu=M_W$ of $O_{1,2}^Z$, O_l^B and $Q^{(6)}$, and the renormalization group evolution down to lower scales, proceeds exactly as in Refs. [5]-[7]. On

[2] For a complete discussion, we refer to Ref. [5]. Note that, since we are interested also in the subleading terms arising by the expansion of the T-products, we include both left-handed and vector components of $Q_Z^{qq\nu\nu}$ in Eq. (7). The latter has been ignored in [5] since it does not contribute to the leading dimension six operator.

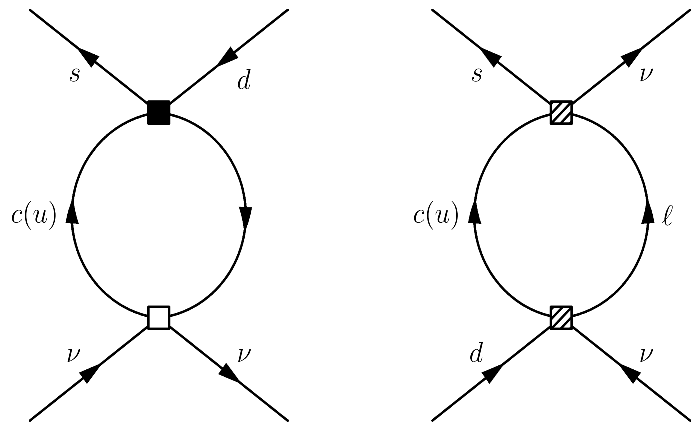

Figure 1: *One-loop diagrams corresponding to the T-products in Eqs. (2)–(4).*

the other hand, we differ from these works in the last step, namely the removal of the charm as dynamical degrees of freedom. In this case we proceed as in Ref. [9], matching the operator product expansion of the T-products into an effective theory which includes also dimension-8 operators. The structure of the local terms, for $\mu_{IR} \lesssim m_c$, takes the form of the following effective Hamiltonian density

$$\mathcal{H}_{eff}^{(6+8)}(\mu_{IR}) = \frac{G_F}{\sqrt{2}} \frac{\alpha}{2\pi \sin^2\theta_W} \lambda_c \sum_{l=e,\mu,\tau} \left[X_c^l(x_c) Q_l^{(6)} + \frac{1}{M_W^2} \sum_i C_i^l(\mu_{IR}) Q_{il}^{(8)} \right] . \quad (9)$$

Neglecting neutrino masses, the only $Q_{il}^{(8)}$ with non-vanishing coefficients to lowest order in $\alpha_s(m_c)$ are

$$\begin{aligned} Q_{1l}^{(8)} &= \bar{s}\gamma^\mu(1-\gamma_5)d\ \partial^2\left[\bar{\nu}_l\gamma_\mu(1-\gamma_5)\nu_l\right] , \\ Q_{2l}^{(8)} &= (\bar{s}\overleftarrow{D}_\alpha)\gamma^\mu(1-\gamma_5)(\overrightarrow{D}_\alpha d)\ \bar{\nu}_l\gamma_\mu(1-\gamma_5)\nu_l , \\ Q_{3l}^{(8)} &= (\bar{s}\overleftarrow{D}_\alpha)\gamma^\mu(1-\gamma_5)d\ \left[\bar{\nu}_l(\overleftarrow{\partial^\alpha} - \overrightarrow{\partial^\alpha})\gamma_\mu(1-\gamma_5)\nu_l\right] . \end{aligned} \quad (10)$$

The operator $Q_{1l}^{(8)}$ arises by the neutral-current coupling (left diagram in Figure 1), while $Q_{2l}^{(8)}$ and $Q_{3l}^{(8)}$ are generated by the charged-current coupling (right diagram in Figure 1). The operator $Q_{3l}^{(8)}$, which has been considered first in Ref. [8], is the only term which can induce a CP-conserving contribution to the $K_2 \to \pi^0\nu_l\bar{\nu}_l$ transition. In agreement with the results of Ref. [8, 9], we find

$$\begin{aligned} C_1^l(\mu_{IR}) &= \frac{1}{12}\left(1 - \frac{4}{3}\sin^2\theta_W\right)\log\left(m_c^2/\mu_{IR}^2\right)\left[3C_1(\mu_c) + C_2(\mu_c)\right] \\ C_2^{e,\mu}(\mu_{IR}) &= \frac{1}{2}\log\left(m_c^2/\mu_{IR}^2\right)\ C_B(\mu_c) \\ C_2^\tau(\mu_{IR}) &= -\frac{1}{4} f\left(m_c^2/m_\tau^2\right)\ C_B(\mu_c) \\ C_3^l(\mu_{IR}) &= -C_2^l(\mu_{IR}) \end{aligned} \quad (11)$$

where $C_{1,2}(\mu_c)$ represent the Wilson coefficients at scale $\mu_c = \mathcal{O}(m_c)$ for the operators in Eqs. (2)–(3), and

$$f(x) = \left(\frac{6x-2}{(x-1)^3} - 2\right)\log x - \frac{4x}{(x-1)^2}\,. \tag{12}$$

In the calculation of the subleading dimension-8 operators, we shall take into account QCD corrections only up to the leading logarithmic level. In this approximation, the $C_{1,2,B}(\mu_c)$ coefficients reads

$$\begin{aligned}
C_1(\mu_c) &= \frac{1}{2}\left[\left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{-6/25}\left(\frac{\alpha_s(m_b)}{\alpha_s(M_W)}\right)^{-6/23} - \left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{12/25}\left(\frac{\alpha_s(m_b)}{\alpha_s(M_W)}\right)^{12/23}\right] \\
C_2(\mu_c) &= \frac{1}{2}\left[\left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{-6/25}\left(\frac{\alpha_s(m_b)}{\alpha_s(M_W)}\right)^{-6/23} + \left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{12/25}\left(\frac{\alpha_s(m_b)}{\alpha_s(M_W)}\right)^{12/23}\right] \\
C_B(\mu_c) &= 1
\end{aligned} \tag{13}$$

Before leaving this section, we comment on the role of the scale μ_{IR}. This scale works as an infrared cutoff for the expansion of the T-products in Eqs. (2)–(4): μ_{IR} separates the long-distance contributions associated to up-quark loops (with low virtuality), from the local part encoded in $\mathcal{H}_{eff}^{(6+8)}(\mu_{IR})$. In order to cancel the μ_{IR} dependence in the physical amplitudes, we should sum to $\langle \pi\nu\bar{\nu}|\mathcal{H}_{eff}^{(6+8)}(\mu_{IR})|K\rangle$ also the non-local contribution generated by the matrix elements of the five four-fermion operators in Eqs. (6)–(8), with $q = u$. In these matrix elements μ_{IR} should act as ultraviolet cut-off for the light degrees of freedom. The estimate of these matrix elements will be addressed in section 3.

2.1 Matrix elements of the dimension-8 operators

The contributions of the dimension-8 operators can be conveniently normalized in terms of the leading matrix element of $Q_l^{(6)}$. The simplest case is the one of $Q_{1l}^{(8)}$, for which we can write

$$\langle \pi^+(k)\nu_l\bar{\nu}_l|Q_{1l}^{(8)}|K^+(p)\rangle = -q^2\ \langle \pi^+\nu_l\bar{\nu}_l|Q_l^{(6)}|K^+\rangle \tag{14}$$

where $q = (p-k)^2 = (p_\nu + p_{\bar{\nu}})^2$.

Concerning $Q_{2l}^{(8)}$ and $Q_{3l}^{(8)}$, we can proceed as in Ref. [8] finding a suitable chiral representation for the corresponding bilinear quark currents. The contribution of $Q_{3l}^{(8)}$, which describes the transition into a $|\nu\bar{\nu}\rangle$ final state with $J = 2$, turns out to be completely negligible [8]. This matrix element i) suffers of a severe kinematical suppression; ii) vanishes to lowest order in the chiral expansion; iii) does not interfere with the leading term. On the contrary, $Q_{2l}^{(8)}$ generates a non-negligible contribution; however, this cannot be expressed in terms of known low-energy couplings. In general, we can write

$$\langle \pi^+(k)\nu_l\bar{\nu}_l|Q_{2l}^{(8)}|K^+(p)\rangle = \hat{B}_2\left[p\cdot k + O(m_q)\right]\ \langle \pi^+\nu_l\bar{\nu}_l|Q_l^{(6)}|K^+\rangle \tag{15}$$

where $\hat{B}_2$ is an unknown hadronic parameter, expected to be of $O(1)$, and $O(m_q)$ denotes contributions proportional to light-quark masses.

In summary, the leading contributions to the $K^+ \to \pi^+ \nu\bar{\nu}$ amplitude generated by the effective Hamiltonian in (9) can be written as

$$\mathcal{A}^{(8)}(K^+ \to \pi^+ \nu_l \bar{\nu}_l) = -\langle \pi^+ \nu_l \bar{\nu}_l | \mathcal{H}^{(6+8)}_{eff}(\mu_{IR}) | K^+ \rangle \equiv \mathcal{A}^{(6)}_Z + \mathcal{A}^{(8)}_Z + \mathcal{A}^{(8)}_{WW} \tag{16}$$

where, adopting the standard CHPT convention $\langle \pi^+ | \bar{s}\gamma^\mu d | K^+ \rangle = -(p+k)^\mu$ and using $\lambda = -\lambda_c$, we have

$$\begin{aligned}
\mathcal{A}^{(6)} &= -\frac{G_F}{\sqrt{2}} \frac{\alpha\ \lambda}{2\pi \sin^2\theta_W} X^l_c(x_c)\ \left[(p+k)^\mu\ \nu_l \gamma_\mu (1-\gamma_5) \nu_l\right] && (17)\\
\mathcal{A}^{(8)}_Z &= \frac{G_F}{\sqrt{2}} \frac{\alpha\ \lambda}{2\pi \sin^2\theta_W} \frac{q^2}{M_W^2} C^l_1(\mu_{IR})\ \left[(p+k)^\mu\ \nu_l \gamma_\mu (1-\gamma_5) \nu_l\right] && (18)\\
\mathcal{A}^{(8)}_{WW} &= -\frac{G_F}{\sqrt{2}} \frac{\alpha\ \lambda}{2\pi \sin^2\theta_W} \hat{B}_2 \frac{p \cdot k}{M_W^2} C^l_2(\mu_{IR})\ \left[(p+k)^\mu\ \nu_l \gamma_\mu (1-\gamma_5) \nu_l\right] && (19)
\end{aligned}$$

3 $K \to \pi\nu\bar{\nu}$ amplitudes within CHPT

As discussed in the previous section, the scale dependence of $K \to \pi\nu\bar{\nu}$ amplitudes induced by the dimension-8 operators must be compensated by a corresponding scale dependence of their long-distance component. The latter is generated by the matrix elements of four-fermion operators which involve only light quarks (u,d and s) and light lepton fields. In this case both internal and external fields do not involve high-energy scales, thus a partonic calculation of this part of the amplitude would be inadequate. In the following we shall present an estimate of these contributions in the framework of CHPT.

The four-fermion operators we are interested in are four-quark operators of the type in Eq. (6) as well as quark-lepton couplings of the type in Eqs. (7)–(8). All these effective operators are generated by the exchange of a single heavy gauge boson (Z or W), and correspondingly have an effective coupling of $O(G_F)$. However, our final goal is the evaluation of their T-product between $|K\rangle$ and $|\pi\nu\bar{\nu}\rangle$ states which –by construction– is of $O(G_F^2)$. As we shall show in the next subsection, this observation has important consequences in the framework of CHPT. In particular, it implies that a consistent treatment of these effects requires the introduction of new appropriate chiral operators of $O(G_F^2 p^2)$.

3.1 The $O(p^2)$ chiral Lagrangian including $O(G_F^2)$ FCNCs

We start by the considering the chiral realization of the $O(G_F)$ coupling between quark and lepton currents. To this purpose, we introduce the so-called strong chiral Lagrangian of $O(p^2)$ in presence of external currents [15][3]

$$\mathcal{L}^{(2)}_S = \frac{F^2}{4} \left\langle D_\mu U D^\mu U^\dagger \right\rangle + \frac{F^2 B}{2} \left\langle \mathcal{M} U + U^\dagger \mathcal{M} \right\rangle\ . \tag{20}$$

[3] The symbol $\langle\rangle$ denotes the trace over the 3×3 flavour space.

As usual, we define

$$U = \exp(\sqrt{2} i\Phi/F) \ , \qquad \Phi = \begin{bmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{bmatrix} \tag{21}$$

where $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$ and, to lowest order, we can identify F with the pion decay constant ($F \approx 92\,\mathrm{MeV}$) and express B in term of meson masses [$m_\pi^2 = B(m_u + m_d)$]. The generic covariant derivative, $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$, allows to systematically include the coupling to external currents (l_μ and r_μ) transforming as $(8_L, 1_R)$ and $(1_L, 8_R)$ under $SU(3)_R \times SU(3)_L$. In the specific case of the $O(G_F)$ couplings to charged lepton currents, we can thus identify the covariant derivative with

$$D_\mu^{(W)} U = \partial_\mu U - i\frac{g}{\sqrt{2}} U (T_+ W_\mu^+ + \mathrm{h.c.}) \ , \qquad T_+ = \begin{pmatrix} 0 & V_{ud} & V_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{22}$$

The situation is slightly more complicated in the case of the Z boson, which couples to both right- and left-handed $SU(3)$ currents, but also to the following singlet current

$$J_{L\mu}^{(1)} = \sum_{q=u,d,s} \bar{q}_L \gamma_\mu q_L \tag{23}$$

which has a non-vanishing $U(1)_L$ charge. As pointed out in [12], the Z coupling to $J_{L\mu}^{(1)}$ involve a new effective coupling which cannot be determined within the $SU(3)_R \times SU(3)_L$ chiral group. Putting all the ingredients together, the covariant derivative with *external* W and Z fields reads

$$D_\mu^{(W,Z)} U = \partial_\mu U - i g_Z Z_\mu \left(\sin^2\theta_W \, [Q, U] + UQ - \frac{a_1}{6} U \right) - i \frac{g}{\sqrt{2}} U (T_+ W_\mu^+ + \mathrm{h.c.}) \ , \tag{24}$$

where $Q = \mathrm{diag}(2/3, -1/3, -1/3)$, $g_Z = g/\cos\theta_W$ and a_1 is the new coupling related to the $U(1)_L$ current.[4] We finally recall that, being external fields, the W^μ and Z^μ operators appearing in Eq. (24) are only a short-hand notation to denote the corresponding leptonic currents (as obtained after the integration of the heavy gauge boson). In particular, the case we are interested in corresponds to

$$W_\mu^+ \to \frac{g}{2\sqrt{2} M_W^2} \sum_l \ \bar{l} \gamma_\mu (1 - \gamma_5) \nu_l \ , \qquad Z_\mu \to \frac{g_Z}{4 M_Z^2} \sum_l \bar{\nu}_l \gamma_\mu (1 - \gamma_5) \nu_l \ . \tag{25}$$

We shall now proceed by considering the $O(G_F)$ four-quark operators. The chiral realization of the $|\Delta S = 1|$ non-leptonic Hamiltonian has been widely discussed in the literature (see e.g. Ref. [16]). The contraction of a single W field coupled to quark currents

[4] The normalization of the $U(1)_L$ coupling is such that $a_1 \to 1$ in the limit where we can extend the symmetry of the QCD action (with 3 massless quarks) from $SU(3)_R \times SU(3)_L$ to $U(3)_R \times U(3)_L$, as it happens for $N_c \to \infty$.

leads to two independent structures which transform as $(8_L, 1_R)$ and $(27_L, 1_R)$, respectively. Given the strong phenomenological suppression of the $(27_L, 1_R)$ terms, in the following we shall consider only the $(8_L, 1_R)$ operators. To lowest order in the chiral expansion, $O(p^2)$, the $(8_L, 1_R)$ non-leptonic weak Lagrangian contains only one term:

$$\mathcal{L}^{(2)}_{|\Delta S|=1} = G_8 F^4 \left\langle \lambda_6 D^\mu U^\dagger D_\mu U \right\rangle \tag{26}$$

Here $G_8 = O(G_F)$ is the effective coupling which is usually fixed from $K \to 2\pi$ and incorporates the phenomenological $\Delta I = 1/2$ enhancement ($G_8 \approx 9 \times 10^{-6}$ GeV^{-2}), while $(\lambda_6)_{ij} = \delta_{2i}\delta_{j3} + \delta_{3i}\delta_{j2}$. As shown in Ref. [17, 18], the number of independent operators increase substantially at $O(p^4)$.

The inclusion in the non-leptonic chiral Lagrangian of external $SU(3)_R \times SU(3)_L$ currents –which are hidden in the covariant derivative in Eq. (26)– allows to describe in a systematic way also non-leptonic weak interactions in presence of external gauge fields. This is for instance the case of the $K \to \pi\gamma$ amplitude analysed in Ref. [19]. However, contrary to what stated in Ref. [12], this minimal coupling is not sufficient in the $K \to \pi\nu\bar{\nu}$ case. Here we are interested in FCNC amplitudes where the leptonic current is associated to a broken generator of the electroweak gauge group: at $O(G_F^2)$ this coupling is not anymore protected by gauge invariance. This argument can easily be understood by looking at the effective Hamiltonian in Eq. (1): from the point of view of chiral symmetry, the FCNC dimension-6 operator in Eq. (1) corresponds to a non-gauge-invariant coupling between a $(8_L, 1_R)$ quark current and an external neutral left-handed current. We thus need to extend the basis of chiral operators and include the chiral realization of all the $O(G_F^2)$ independent terms of this type. At $O(p^2)$ the situation is again quite simple since we have only two independent terms:

$$\left\langle \lambda_6 U^\dagger D^\mu U l_\mu \right\rangle \ , \qquad\qquad \left\langle \lambda_6 U^\dagger D^\mu U \right\rangle \langle l_\mu \rangle \ . \tag{27}$$

In the specific case we are interested in, we can thus add to the $|\Delta S| = 1$ Lagrangian in Eq. (26), with the covariant derivative (24), the following $O(G_F^2 p^2)$ term

$$\mathcal{L}^{(2)}_{\rm FCNC} = i g_Z F^4 Z_\mu \left[G_8^Z \left\langle \lambda_6 U^\dagger D^\mu U Q \right\rangle + G_1^Z a_1 \left\langle \lambda_6 U^\dagger D^\mu U \right\rangle \right] \ , \tag{28}$$

where again the Z^μ field has to be understood as the neutral current in Eq. (25).

The consistent inclusion of all the relevant $O(G_F^2 p^2)$ operators has forced us to introduce two new effective chiral couplings of $O(G_F)$, namely G_8^Z and G_1^Z. At this order the chiral Lagrangian $\mathcal{L}^{(2)}_{\rm FCNC} + \mathcal{L}^{(2)}_{|\Delta S|=1}$ –with the covariant derivative given in (24)– has a local current-current structure identical to the one of the leading dimension-6 Hamiltonian in Eq. (1). Since we are able to compute explicitly the $K^0 \to Z$ matrix element in both approaches, we can fix the unknown chiral couplings by an appropriate matching condition on the $K^0 \to Z$ amplitude. In particular, imposing the condition

$$\langle K^0 | \mathcal{L}^{(2)}_{|\Delta S|=1} + \mathcal{L}^{(2)}_{\rm FCNC} | Z^\mu \rangle = 0 \tag{29}$$

we eliminate from the $O(G_F^2 p^2)$ chiral Lagrangian any contamination from the leading dimension-6 operators. The matrix elements of the latter must then be computed directly

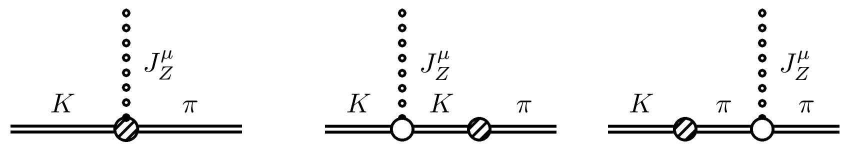

Figure 2: *Tree-level contributions to the $K \to \pi Z$ amplitudes within CHPT. Dashed and empty circles correspond to vertices derived from $\mathcal{L}^{(2)}_{|\Delta S|=1+\mathrm{GIM}}$ and $\mathcal{L}^{(2)}_S$, respectively*

by means of the partonic Hamiltonian, as in the standard approach. The condition (29) implies $G_8^Z = -2G_8$ and $G_1^Z = G_8/3$, which leads to the following structure for the weak Lagrangian of $O(G_F p^2 + G_F^2 p^2)$:

$$\mathcal{L}^{(2)}_{|\Delta S|=1+\mathrm{GIM}} = G_8 F^4 \langle \lambda_6 \left[D^\mu U^\dagger D_\mu U - 2ig_Z Z_\mu U^\dagger D^\mu U \left(Q - \frac{a_1}{6} \right) \right] \rangle \qquad (30)$$

Using this Lagrangian,[5] together with the ordinary $\mathcal{L}^{(2)}_S$, we are finally ready to analyse the structure of $K \to \pi\nu\bar{\nu}$ long-distance amplitudes up to the one-loop level.

3.2 Z-mediated amplitudes: general structure and $O(p^2)$ results

Following the same conventions adopted for the short-distance calculation, we define the $K(p) \to \pi(k)\nu\bar{\nu}$ amplitude mediated by Z exchange by

$$\mathcal{A}(K \to \pi\nu\bar{\nu})_Z = -i\langle \pi | \frac{\delta}{\delta Z_\mu} \int [d\phi] e^{i\int dx \mathcal{L}[\phi, Z]} \, |K\rangle \times \frac{g_Z}{4M_Z^2} \sum_l \bar{\nu}_l \gamma_\mu (1-\gamma_5)\nu_l \qquad (31)$$

and we find it convenient to decompose it as follows

$$\mathcal{A}(K^i \to \pi^i \nu\bar{\nu})_Z = \frac{G_F}{\sqrt{2}} G_8 F^2 \left[\mathcal{M}^i_L \, p^\mu + \mathcal{M}^i_V \left(q^2 p^\mu - p\cdot q \; q^\mu \right) \right] \sum_l \bar{\nu}_l \gamma_\mu (1-\gamma_5)\nu_l \qquad (32)$$

where $q = p - k$ and the form factors $\mathcal{M}^i_{V,L}$ are regular functions in the limit $q^2 \to 0$ and $p \cdot q \to 0$. The $\mathcal{M}^i_V$ terms, which can be different from zero only at $O(p^4)$ in the chiral expansion, correspond to the conserved component of the coupling to the external current.

In the decomposition (32) we have implicitly neglected the $O(q^\mu)$ terms which do not appear in the coefficient of $\mathcal{M}^i_V$ (all the $O(q^\mu)$ terms give a negligible result when contracted with the neutrino current). This allows to substantially simplify the calculation and, in particular, to neglect the bilinear couplings $Z^\mu \partial_\mu \phi$. In this limit, the only non-vanishing $O(p^2)$ tree-level diagrams contributing the $K \to \pi Z$ amplitude are those in

[5] We denote it with a subscript "GIM" since the condition (29) is equivalent to enforcing an exact GIM cancellation for the tree-level FCNC associated to the leading short-distance operator.

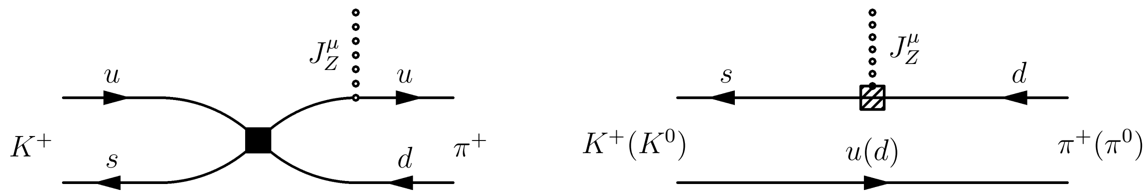

Figure 3: *Examples of valence-quark topologies appearing in $K \to \pi Z$ amplitudes. Left: non-spectator topology contributing to the $K^+ \to \pi^+$ transition only (the black box denotes a weak four-quark operator). Right: penguin-type contraction or insertion of the local FCNC effective coupling.*

figure 2. They leads to the following results for the non-conserved terms of charged and neutral channels:

$$\mathcal{M}_L^{+(2)} = 4 \ , \qquad \mathcal{M}_L^{0(2)} = 0 \ . \tag{33}$$

Because of the modified weak Lagrangian in (30), these results are qualitatively different from those available in the literature. On the one side, find that the charged amplitude ($K^+ \to \pi^+ Z$) is different from zero and is completely determined in terms of known couplings. In particular, the amplitude does not vanish in the large N_C limit, as claimed in Ref. [12]. On the other hand, we find that the neutral amplitude ($K^0 \to \pi^0 Z$) is identically zero. By comparison, it should be noted that the $K^0 \to \pi^0 Z$ amplitude computed with the minimal-coupling prescription of Ref. [12] is different from zero, even in the large N_C limit. The vanishing of the $K^0 \to \pi^0 Z$ amplitude at $O(p^2)$ is a direct consequence of the condition imposed by Eq. (29), which removes from the low-energy effective Lagrangian the spurious tree-level FCNC coupling generated by the minimal substitution. As already mentioned, by means of this procedure the leading penguin-type contractions (see figure 3) are removed from the low-energy effective theory and treated directly by means of the partonic Hamiltonian. In this approach, the only non-vanishing contributions of $O(p^2)$ are the genuine long-distance effects associated to non-spectator topologies, which can affect only the charged transition $K^+ \to \pi^+ Z$ (see figure 3).

3.3 One-loop contributions to $K \to \pi Z$ amplitudes

The $O(p^4)$ calculation of the $K \to \pi Z$ amplitude involves several one-loop diagrams. However, a substantial simplification is obtained by performing the calculation in the basis of Ref. [20], where the weak $O(p^2)$ mixing among pseudoscalar mesons is diagonalized, and neglecting the $O(q^\mu)$ pole diagrams due to the $Z^\mu \partial_\mu \phi$ coupling (as at the tree-level, this completely removes the unknownw a_1 parameter from the relevant vertices). With these simplifications, the relevant one-loop diagrams are shown in figure 4.

In the case of the $\mathcal{M}_L^+$ form factor, the complete one-loop contribution can be decomposed as

$$\mathcal{M}_L^{+(4)}\Big|_{\rm loop} = \frac{2}{(4\pi F)^2}\left[\mathcal{T}^0_{\rm a+b+c} - (m_K^2 - m_\pi^2)\mathcal{T}^1_{\rm a+b+c}\right] + \frac{1}{2}\left(\delta Z_K + \delta Z_\pi\right)\mathcal{M}_L^{+(2)} \tag{34}$$

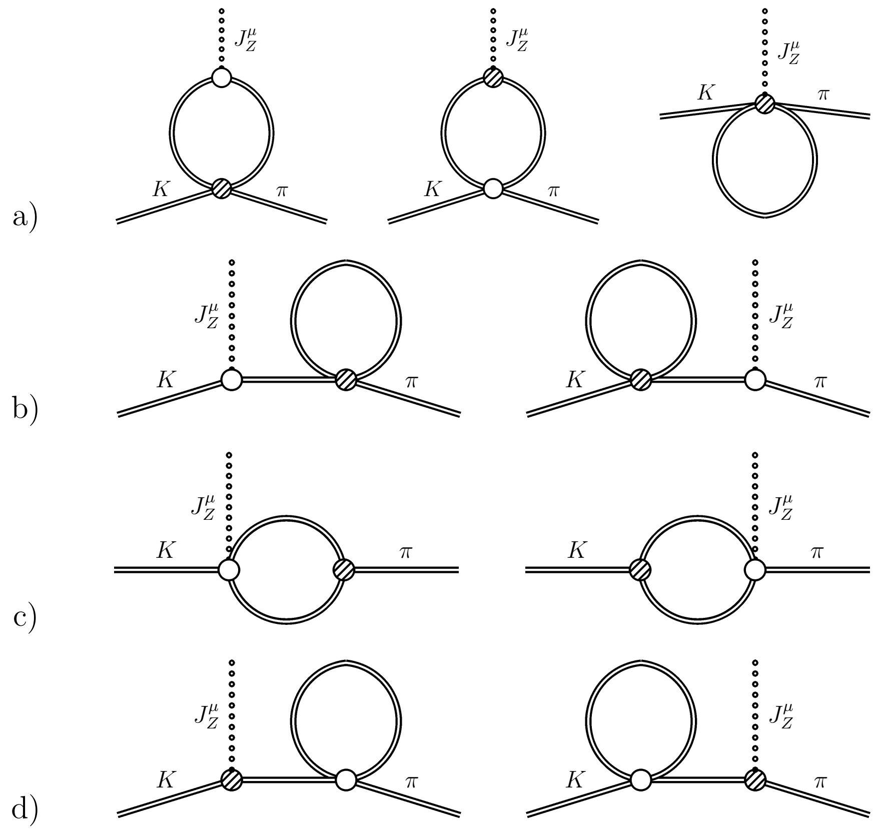

Figure 4: *One-loop contributions to the $K \to \pi Z$ amplitudes within CHPT in the pseudoscalar basis of Ref. [20]. Dashed and empty circles correspond to flavour-changing and flavour-conserving vertices, respectively. The wave-function renormalization diagrams (d) contribute only to $\mathcal{M}_L^+$ (which is non-zero already at lowest order).*

where the subscripts refer to the labels of the diagrams in figure 4. The explicit computation of the various terms yields

$$\begin{aligned}\mathcal{T}^0_{\rm a+b+c} &= A_0\left(m_K^2\right) + A_0\left(m_{\eta_8}^2\right) - \frac{1}{6}\left(4m_K^2 - q^2\right) B_0\left(q^2; m_K^2, m_\pi^2\right) \\ &\quad + \frac{1}{12}\left(4m_K^2 - q^2\right) B_0\left(q^2; m_K^2, m_K^2\right) + \frac{1}{12}\left(4m_\pi^2 - q^2\right) B_0\left(q^2; m_\pi^2, m_\pi^2\right) \end{aligned} \tag{35}$$

$$\begin{aligned}\mathcal{T}^1_{\rm a+b+c} &= \frac{1}{6}\left(2 + \frac{m_K^2}{m_\pi^2}\right) B_0\left(m_\pi^2; m_K^2, m_{\eta_8}^2\right) + \frac{1}{6}\left(1 - \frac{m_K^2}{m_\pi^2}\right) B_0\left(0; m_K^2, m_{\eta_8}^2\right) \\ &\quad + \left(\frac{2}{3} - \frac{m_K^2}{2m_\pi^2} + \frac{m_K^2 - m_\pi^2}{6q^2}\right) B_0\left(0; m_K^2, m_\pi^2\right) \\ &\quad - \left(\frac{1}{3} + \frac{m_K^2 - m_\pi^2}{6q^2}\right) B_0\left(q^2; m_K^2, m_\pi^2\right) + \frac{m_K^2}{2m_\pi^2} B_0\left(m_\pi^2; m_K^2, m_\pi^2\right) \end{aligned} \tag{36}$$

$$\delta Z_\pi = -\frac{1}{(4\pi F)^2}\left[\frac{2}{3}A_0\left(m_\pi^2\right)+\frac{1}{3}A_0\left(m_K^2\right)\right] \tag{37}$$

$$\delta Z_K = -\frac{1}{(4\pi F)^2}\left[\frac{1}{4}A_0\left(m_\pi^2\right)+\frac{1}{2}A_0\left(m_K^2\right)+\frac{1}{4}A_0\left(m_{\eta_8}^2\right)\right] \tag{38}$$

where the expressions for the loop functions A and B, evaluated in dimensional regularization, are given in the Appendix.

Interestingly, the complete expression of $\mathcal{M}_L^{+(4)}|_{\rm loop}$ is finite and vanishes exactly in the $SU(3)$ limit ($m_K = m_{\eta_8} = m_\pi$). The amplitude has a mild q^2 dependence and for $q^2 > 4m_\pi^2$ develops a small imaginary part –related to the $K \to 3\pi$ intermediate state– via the loop function $B_0(q^2; m_\pi^2, m_\pi^2)$. From the numerical point of view, the various terms in $\mathcal{M}_L^{+(4)}|_{\rm loop}$ are separately of $O(1)$; however, there is a strong cancellation among them: using the Gell-Mann Okubo (GMO) relation among pseudoscalar masses, we find

$$\left|\mathcal{M}_L^{+(4)}\right|_{\rm loop} < 0.01 \tag{39}$$

in all the allowed q^2 range. This cancellation is very sensitive to possible violations of the GMO relation, but even for physical masses we find a result substantially smaller than the tree-level value in Eq. (33).

In the case of the $\mathcal{M}_V^+$ form factor, the result of the one-loop calculation yields

$$\begin{aligned}\mathcal{M}_V^{+(4)}\Big|_{\rm loop} = \frac{2}{(4\pi F)^2}\left(1-\frac{4}{3}\sin^2\theta_W\right)\Bigg[&\frac{m_K^2}{q^2}B_0\left(0;m_K^2,m_K^2\right)+\frac{m_\pi^2}{q^2}B_0\left(0;m_\pi^2,m_\pi^2\right)\\ &+\frac{1}{4}B_0\left(q^2;m_K^2,m_K^2\right)\left(1-\frac{4m_K^2}{q^2}\right)+\frac{1}{4}B_0\left(q^2;m_\pi^2,m_\pi^2\right)\left(1-\frac{4m_\pi^2}{q^2}\right)+\frac{1}{3}\Bigg]\end{aligned} \tag{40}$$

As expected, this result coincides (up to the overall normalization) with the one-loop expression of the $K^+ \to \pi^+\gamma$ amplitude obtained by Ecker, Pich and de Rafael [19]. To make more explicit the connection with their result, it is sufficient to note that

$$\mathcal{M}_V^{+(4)}\Big|_{\rm loop} = \frac{1}{(4\pi F)^2}\left(1-\frac{4}{3}\sin^2\theta_W\right)\left[D_\varepsilon - \log\frac{m_K m_\pi}{\mu^2}+3\phi_\pi\left(q^2\right)+3\phi_K\left(q^2\right)\right] \tag{41}$$

where D_ε and $\phi_i(q^2)$ –defined as in Ref. [19]– are reported in the Appendix. Contrary to the one-loop expression of $\mathcal{M}_L^+$, the result in (40) is not finite and does not vanish in the $SU(3)$ limit.

Although not strictly necessary for practical purposes, we report here also the one-loop results for the neutral form factors, which are useful to investigate the $SU(3)$ properties of $K \to \pi Z$ amplitudes. Due to the absence of tree-level contributions, the calculation of

the neutral form factors is somewhat simpler. We find

$$\mathcal{M}_L^{0(4)}\Big|_{\text{loop}} = -\frac{\sqrt{2}}{(4\pi F)^2}\Big\{-\frac{5}{3}\left[A_0\left(m_K^2\right)-A_0\left(m_\pi^2\right)\right]$$
$$+\frac{1}{6}\left(4m_K^2-q^2\right)\left[B_0\left(q^2;m_K^2,m_K^2\right)-B_0\left(q^2;m_K^2,m_\pi^2\right)\right]$$
$$+\left(m_K^2-m_\pi^2\right)\Big[\frac{1}{3}-2B_0\left(m_K^2;m_\pi^2,m_\pi^2\right)+\frac{m_K^2-m_\pi^2+2q^2}{6q^2}B_0\left(q^2;m_K^2,m_\pi^2\right)$$
$$-\frac{m_K^2-m_\pi^2+q^2}{6q^2}B_0\left(0;m_K^2,m_\pi^2\right)\Big]\Big\} \tag{42}$$

and

$$\mathcal{M}_V^{0(4)}\Big|_{\text{loop}} = -\frac{1}{\sqrt{2}(4\pi F)^2}\left(1-\frac{4}{3}\sin^2\theta_W\right)\left[D_\varepsilon-\log\frac{m_K^2}{\mu^2}+6\phi_K\left(q^2\right)\right] \tag{43}$$

Also in this case the $\mathcal{M}_L$ term vanishes in the $SU(3)$ limit and there is perfect analogy with the result of Ref. [19] for the vector form factor. However, contrary to the charged case, Eq. (42) is not finite beyond the $SU(3)$ limit. In particular, we find

$$\mathcal{M}_L^{0(4)}\Big|_{\text{loop}}^{\text{div}} = \frac{7(m_K^2-m_\pi^2)}{\sqrt{2}(4\pi F)^2}D_\varepsilon \tag{44}$$

Concerning the finite parts, the one-loop expression of $\mathcal{M}_L$ is $O(1)$, it is almost constant, and it has a large absorptive part associated to the $K^0\to\pi^+\pi^-$ intermediate state.

Combining all the one-loop results, the $SU(3)$ limit of the two amplitudes satisfy the relation

$$\mathcal{A}(K^+\to\pi^+\nu\bar{\nu})_Z^{\text{loop}}\Big|_{SU(3)} = -\sqrt{2}\,\mathcal{A}(K^0\to\pi^0\nu\bar{\nu})_Z^{\text{loop}}\Big|_{SU(3)} \equiv \mathcal{A}_Z^{\text{loop}}\ , \tag{45}$$

in perfect analogy with the $K^+\to\pi^+\gamma$ case. In this limit, the (UV) scale dependence of the amplitude is given by

$$\mu^2\frac{d}{d\mu^2}\mathcal{A}_Z^{\text{loop}} = \left(1-\frac{4}{3}\sin^2\theta_W\right)\frac{G_F G_8}{16\sqrt{2}\pi^2}\times q^2\left[p^\mu\nu_l\gamma_\mu(1-\gamma_5)\nu_l\right]\ . \tag{46}$$

3.4 Matching and $O(p^4)$ counterterms for $K\to\pi Z$ amplitudes

One of the key features of our result, is the fact that the scale dependence of the one-loop CHPT amplitude turns out to be proportional to $(1-\frac{4}{3}\sin^2\theta_W)$. This fact is particularly welcome since it signals a short-distance behavior in agreement with the one derived at the partonic level. Indeed, the same coupling appears in the Wilson coefficient of $Q_{1l}^{(8)}$, namely the four-fermion dimension-8 operator corresponding to the Z-penguin contraction. Thus the short-distance behavior of the one-loop CHPT amplitude is perfectly compatible with

the IR structure derived by the partonic calculation. To be more explicit, the UV scale dependence in (46) should be compared with

$$\mu_{IR}^2 \frac{d}{d\mu_{IR}^2} \mathcal{A}_Z^{(8)} = -\left(1 - \frac{4}{3}\sin^2\theta_W\right) \frac{G_F^2 \lambda [3C_1(\mu_c) + C_2(\mu_c)]}{12\pi^2} \times q^2 \left[p^\mu \nu_l \gamma_\mu (1-\gamma_5)\nu_l\right], \quad (47)$$

which follows from Eq. (19). As expected, the two expressions do not match exactly, given the non-perturbative QCD effects encoded in G_8, but they have the same kinematical dependence, and the same parametrical dependence from the electroweak couplings. The second feature is a direct consequence of the matching condition which has been imposed on the $O(G_F^2 p^2)$ Lagrangian in (30).

As a consequence of the non-perturbative enhancement of G_8 ($G_8 \gg G_F\lambda$), which is a manifestation of the $\Delta I = 1/2$ rule, the scale dependence derived within CHPT is substantially larger with respect to the one obtained by the dimension-8 partonic Hamiltonian. This fact signals that, at least in the case of Z-mediated amplitudes, the genuine long-distance contributions evaluated within CHPT represent the dominant effect.

Within a pure CHPT approach, the scale dependence of the $\mathcal{L}^{(2)} \times \mathcal{L}^{(2)}$ one-loop amplitude must be exactly compensated by that of the $O(p^4)$ local counterterms, whose renormalized finite parts encode possible short-distance contributions. In the present case, we can distinguish three types of $O(p^4)$ counterterms:

i. the $L_i(\mu)$ of the strong Lagrangian [15];

ii. the $N_i(\mu)$ of the $O(G_F p^4)$ non-leptonic weak Lagrangian [18];

iii. new $O(G_F^2 p^4)$ local terms with explicit l_μ fields, namely the $O(p^4)$ generalization of the $O(p^2)$ terms in Eq. (27).

Unfortunately, only in the first two cases the finite parts of the counterterms are experimentally known. Considering only the contribution of these known couplings, we can write

$$\begin{aligned}
\left.\mathcal{M}_V^{+(4)}\right|_{\rm CT} &= \frac{4}{F^2}\left(1 - \frac{4}{3}\sin^2\theta_W\right)[N_{14}(\mu) - N_{15}(\mu) + 3L_9(\mu)] + O(N_i') \\
&\equiv \frac{1}{(4\pi F)^2}\left(1 - \frac{4}{3}\sin^2\theta_W\right)\left[1 - 3\frac{G_F}{|G_8|}a_+ + \log\frac{m_K m_\pi}{\mu^2}\right] + O(N_i') \qquad (48) \\
\left.\mathcal{M}_V^{0(4)}\right|_{\rm CT} &= -\frac{\sqrt{2}}{F^2}\left(1 - \frac{4}{3}\sin^2\theta_W\right)[2N_{14}(\mu) + N_{15}(\mu)] + O(N_i') \\
&\equiv -\frac{1}{\sqrt{2}(4\pi F)^2}\left(1 - \frac{4}{3}\sin^2\theta_W\right)\left[1 + 3\frac{G_F}{|G_8|}a_S + \log\frac{m_K^2}{\mu^2}\right] + O(N_i') \qquad (49)
\end{aligned}$$

where $O(N_i')$ denote $\sin^2\theta_W$-independent and μ^2-independent combination of counterterms which includes the new unknown couplings, as well as the unknown a_1 parameter from the $O(p^4)$ strong and non-leptonic weak Lagrangian (i. and ii.). According to the

experimental data on $K^+ \to \pi^+\ell^+\ell^-$ decays, the numerical value of the (finite) low-energy constant appearing in (48) is $a_+ \approx -0.6$ [21, 22].

Concerning $\mathcal{M}_L$, the counterterm structure is more complicated. In the charged mode, where the tree-level result is different from zero, we expect terms correcting G_8 and the meson decay constants. From the konwn size of the latter, we naturally expect $O(25\%)$ corrections to the tree-level results, though we cannot exclude that strong cancellations similar to those leading to Eq. (39) do occur. It should also be noted that spurious $O(G_F^2 p^4)$ effects from dimension-6 operators are still present in the q^2-independent part of both $\mathcal{M}_L^{+(4)}$ and $\mathcal{M}_L^{0(4)}$; however, disentangling these effects from the genuine $O(G_F^2 p^4)$ long-distance corrections is beyond the scope of this work. Given these uncertainties on the counterterm structure of the $\mathcal{M}_L$ terms, in the numerical analysis of section 4 we shall assign a conservative 50% error to the tree-level result for $\mathcal{M}_L^+$.

3.5 W–W amplitudes

As can be seen from figure 5, within CHPT the structure of FCNC amplitudes induced by two charged currents is substantially simpler than the Z-mediated case. On general grounds, since the heavy τ lepton has already been integrated out at the level of the partonic Hamiltonian, we can decompose these contributions to the $K^+ \to \pi^+\nu\bar{\nu}$ amplitude as follows

$$\mathcal{A}(K^+ \to \pi^+\nu\bar{\nu})_{WW} = G_F^2 F^2 \lambda \sum_{l=e,\mu} \mathcal{M}_{WW}^l \; p^\mu \bar{\nu}_l \gamma_\mu (1-\gamma_5)\nu_l \tag{50}$$

The $O(p^2)$ tree-level contribution has already been analysed in Ref. [11]; however, we disagree with their final result by a factor of 4. In particular, we obtain

$$\mathcal{M}_{WW}^{l(2)} = 2\left[1 + \frac{m_l^2}{t - m_l^2}\right] \approx 2 \qquad\qquad (l = e, \mu) \tag{51}$$

where $t = (p - p_\nu)^2$. Given the strong kinematical suppression of terms asymmetric in the two neutrino momenta [8], the approximate result in the r.h.s. of Eq. (51) turns out to be an excellent approximation in the evaluation of the total decay rate.

The $O(p^4)$ one-loop calculation yields

$$\begin{aligned} \mathcal{M}_{WW}^{l(4)}\Big|_{\rm loop} \;=\; & \frac{1}{(4\pi F)^2}\Big\{ (m_\pi^2 - m_l^2)\left(2 - \frac{m_l^2}{2t}\right) B_0(0; m_l^2, m_\pi^2) + 4A_0(m_l^2) - A_0(m_\pi^2) \\ & + \left[4(m_\pi^2 - t) + (m_l^2 - m_\pi^2 + t)\left(2 - \frac{m_l^2}{2t}\right)\right] B_0(t; m_l^2, m_\pi^2)\Big\} \end{aligned} \tag{52}$$

and in the limit $m_l \to 0$ becomes (see Appendix):

$$\begin{aligned} \mathcal{M}_{WW}^{l(4)}\Big|_{\rm loop} \;\stackrel{m_l\to 0}{\longrightarrow}\; & \frac{1}{(4\pi F)^2}\Big[(3m_\pi^2 - 2t)\left(D_\varepsilon - \log\frac{m_\pi^2}{\mu^2}\right) \\ & + 5m_\pi^2 - 4t + 2\frac{(m_\pi^2 - t)^2}{t}\log\left(1 - \frac{t}{m_\pi^2}\right)\Big] \end{aligned} \tag{53}$$

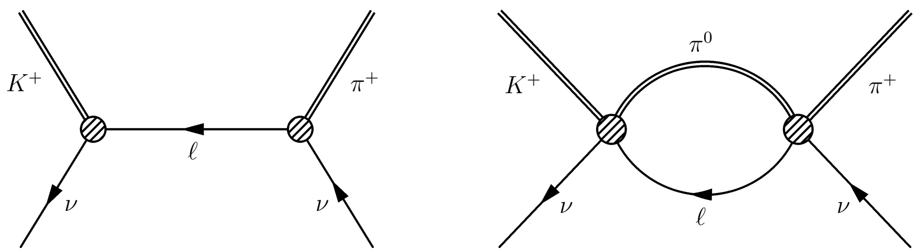

Figure 5: *Tree-level and one-loop contributions to $K^+ \to \pi^+\nu\bar{\nu}$ amplitudes induced by W–W exchange, within CHPT. Contrary to the case of figure 1, here the index ℓ runs only over the first two families of leptons.*

As expected, the one-loop result is divergent. Within a pure CHPT calculation, such divergence is cured by appropriate $O(G_F^2 p^4)$ counterterms. Unfortunately, we cannot determine the finite part of these counterterms, neither from first principles nor from data. However, a reasonable estimate of their size can be obtained by imposing the matching of the scale dependence with the short-distance partonic calculation. Since in this case there are no sizable non-perturbative effects associated to the $\Delta I = 1/2$ rule (as in the Z-mediated case), we expect a good numerical matching. From Eq. (53) it follows

$$\begin{aligned} \mu^2 \frac{d}{d\mu^2} \mathcal{A}_{WW}^{\rm loop} &= \frac{G_F^2 \lambda}{16\pi^2}(3m_\pi^2 - 2t)\,[p^\mu \nu_l \gamma_\mu (1-\gamma_5)\nu_l] \\ &= -\frac{G_F^2\lambda}{8\pi^2}\left[p\cdot k + O(m_\pi^2)\right][p^\mu \nu_l \gamma_\mu(1-\gamma_5)\nu_l] \ + \ O\left[p\cdot(p_\nu - p_{\bar\nu})\right] \ , \qquad (54) \end{aligned}$$

while from Eq. (19) we get

$$\mu_{IR}^2 \frac{d}{d\mu_{IR}^2}\mathcal{A}_{WW}^{(8)} = \frac{G_F^2\lambda}{2\pi^2}\left[\hat{B}_2(p\cdot k) + O(m_q)\right][p^\mu \nu_l\gamma_\mu(1-\gamma_5)\nu_l] \ . \qquad (55)$$

As can be noted, the two expression have the same kinematical structure (within the approximations employed) and the scale dependence can be matched with an appropriate choice of the hadronic parameter defined in Eq. (15), namely $\hat{B}_2 \approx 1/4$. This result allows us to have a full control on the total W–W amplitude, with the inclusion of the contribution of the dimension-8 partonic operator. Neglecting $O(p^4)$ terms not enhanced by the large factor $\log(m_c^2/m_\pi^2)$, we finally obtain

$$\mathcal{M}_{WW}^{(4)} \ = \ 2 - \frac{1}{16\pi^2 F^2}(m_K^2 - q^2)\log\frac{m_c^2}{m_\pi^2} + O\left(\frac{m_K^2}{16\pi^2F^2}\right) \qquad (l = e,\mu) \qquad (56)$$

Note that, despite the large-log enhancement, the $O(p^4)$ term turns out to be smaller than the tree-level contribution. It is also worth to stress that the overall term in (56) is smaller that the estimate of the dimension-8 contribution only presented in Ref. [9], which was based on naïve dimensional analysis.

4 Numerical analysis for $\mathcal{B}(K^+ \to \pi^+ \nu\bar{\nu})$

Collecting the results of the previous section, we can finally estimate the *complete* subleading Z– and W–W–mediated contributions to the $K^+ \to \pi^+\nu\bar{\nu}$ amplitude (including both dimension-8 and long-distance effects). We express them through the coefficients $P_Z(q^2)$ and $P_{WW}(q^2)$, defined by

$$\mathcal{A}(K^+ \to \pi^+\nu\bar{\nu}) = -\frac{G_F}{\sqrt{2}}\frac{\alpha\ \lambda^5}{2\pi\sin^2\theta_W}\sum_{l=e,\mu,\tau}\left[P_Z(q^2)+P^l_{WW}(q^2)\right](p+k)^\mu\bar{\nu}_l\gamma_\mu(1-\gamma_5)\nu_l \quad (57)$$

The normalization of $P_Z(q^2)$ and $P_{WW}(q^2)$ is such that they can easily be compared with the leading dimension-6 charm contribution in Eq. (17). In particular, the combination

$$\delta P_{c,u} = \frac{1}{3}\sum_{l=e,\mu,\tau}\left\langle P_Z(q^2)+P^l_{WW}(q^2)\right\rangle \quad (58)$$

where $\langle\rangle$ denotes an appropriate average over the phase space, represents the correction to be added to the leading coefficient

$$P_c^{(6)} = \frac{1}{\lambda^4}\left[\frac{2}{3}X_c^e+\frac{1}{3}X_c^\tau\right] = 0.39\pm 0.07\ , \quad (59)$$

whose numerical value corresponds to the present NLO accuracy of $\mathcal{H}^{(6)}_{eff}$ [7, 4].

According to the results in Eq. (33), (41) and (48), we find

$$P_Z(q^2) \ = \ -\frac{\pi^2F^2\mathrm{sgn}(G_8)}{\sqrt{2}\lambda^5M_W^2}\left[\frac{4|G_8|}{G_F}-\frac{3a_+q^2}{16\pi^2F^2}\left(1-\frac{4}{3}\sin^2\theta_W\right)+O(N_i',q^4)\right] \quad (60)$$

while for the W–W contribution we get

$$\begin{aligned} P^{e,\mu}_{WW}(q^2) \ &= \ -\frac{\pi^2F^2}{\lambda^4M_W^2}\left[2-\frac{1}{16\pi^2F^2}(m_K^2-q^2)\log\frac{m_c^2}{m_\pi^2}+O\left(\frac{m_K^2}{16\pi^2F^2}\right)\right] \\ P^{\tau}_{WW}(q^2) \ &= \ -\frac{(m_K^2-q^2)}{32\lambda^4M_W^2}f\left(m_c^2/m_\tau^2\right) \end{aligned} \quad (61)$$

In the case of $P_Z(q^2)$, we have explicitly pointed out the dependence from sgn(G_8), which has to be estimated starting from the partonic four-quark Hamiltonian. As discussed in Ref. [22], employing the factorization approximation leads to $G_8<0$. From the numerical point of view, the sum of the $O(p^4)$ terms encoded in $P_Z(q^2)$ and $P_{WW}(q^2)$ turn out to be about 20% of the $O(p^2)$ terms, in good agreement with naïve chiral counting. However, as discussed at the end of section 3.4, we have not been able to estimate all the possible $O(p^4)$ contributions. For this reason, we believe that the most conservative approach for the numerical analysis is obtained by fixing the central value of $\delta P_{c,u}$ from the complete $O(p^2)$ result, and attribute to it a 50% error:

$$\delta P_{c,u} \ \approx \ \frac{\pi^2F^2}{\lambda^4M_W^2}\left[\frac{4|G_8|}{\sqrt{2}\lambda G_F}-\frac{4}{3}\right] \ = \ 0.04\pm 0.02 \quad (62)$$

In summary, the subleading contributions of $O(G_F^2\Lambda_{\rm QCD}^2)$ to the $K^+ \to \pi\nu\bar{\nu}$ amplitude turns out to be a *constructive* 10% correction with respect to the leading charm contribution

$$P_c^{(6)} \to P_c^{(6)} + \delta P_{c,u} \qquad\qquad \delta P_{c,u} = 0.04 \pm 0.02 \tag{63}$$

This correction implies a $\approx 6\%$ increase of the SM prediction of $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$.

The overall effect of the $O(G_F^2\Lambda_{\rm QCD}^2)$ corrections turns out to be smaller than the error of the dimension-6 term at the NLO level, thus it was justified to neglect these subleading terms at this level of accuracy (as for instance done in the recent analysis of Ref. [4, 23]). However, these subleading terms are not negligible in view of a NNLO analysis of the dimension-6 contribution.

5 Conclusions

We have presented a comprehensive analysis of the $O(G_F^2\Lambda_{\rm QCD}^2)$ contributions to $K \to \pi\nu\bar{\nu}$ amplitudes not described by the leading dimension-6 effective Hamiltonian. These include both the effects of dimension-8 four-fermion operators generated at the charm scale, and the genuine long-distance contributions which can be described within the framework of CHPT. As we have shown, these two type of effects are closely correlated. The main results of our analysis can be summarized as follows:

- The dominant contributions are the $O(G_F^2 p^2)$ tree-level amplitudes which can be computed within CHPT. A consistent evaluation of these amplitudes requires the introduction of new chiral operators of $O(G_F^2 p^2)$, in addition to those already present in the non-leptonic weak chiral Lagrangian. These operators, which are needed to cancel spurious tree-level FCNCs and to ensure a correct UV behavior of the chiral amplitudes, have not been considered in the previous literature [12, 13, 14].

- The introduction of the new $O(G_F^2 p^2)$ operators has allowed us to obtain a consistent matching between the CHPT one-loop amplitudes of $O(G_F^2 p^4)$ and the contributions of the dimension-8 four-fermion Hamiltonian. Thanks to this matching, we have been able to estimate more precisely both these sources of subleading contributions. In particular, we have estimated the hadronic matrix element of all the dimension-8 partonic operators of Ref. [9], strongly reducing this source of uncertainty. From the numerical point of view, both the $O(G_F^2 p^4)$ chiral amplitudes and the dimension-8 partonic operators induce very small effects on $K \to \pi\nu\bar{\nu}$ amplitudes (corrections below the 1% level).

- In the $K^+ \to \pi^+\nu\bar{\nu}$ case, the leading corrections due to the $O(G_F^2 p^2)$ chiral amplitudes amount to about 10% of the dimension-6 charm contribution, or about 3% of the total SM amplitude. Their effect can be efficiently encoded in the standard analysis of $\mathcal{B}(K^+ \to \pi^+\nu\bar{\nu})$ be means of Eq. (63). The size of these effects can easily be understood by noting that they scale as $(\pi F/m_c)^2 \approx 5\%$ with respect to the leading charm contribution and are partially enhanced by the $\Delta I = 1/2$ rule (see

Eq. (62)). These subleading terms are not negligible in view of a NNLO analysis of the dimension-6 contribution.

Acknowledgments

We thank Andrzej Buras, Gerhard Ecker, Guido Martinelli, Eduardo de Rafel, and Paolo Turchetti for useful comments and discussions. This work is partially supported by IHP-RTN, EC contract No. HPRN-CT-2002-00311 (EURIDICE).

A Loop functions

Following standard conventions, we define

$$B_0\left(q^2,m_1^2,m_2^2\right) = -i\,(4\pi)^2\,\mu^\varepsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2-m_1^2\right)\left((k-p)^2-m_2^2\right)} \tag{64}$$

$$A_0(m^2) = -i\,(4\pi)^2\,\mu^\varepsilon \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2-m^2\right)} \tag{65}$$

where $\varepsilon = 4-d$. The following identities holds

$$\begin{aligned}
B_0\left(0,m^2,m^2\right) &= \frac{A_0\left(m^2\right)}{m^2} - 1 = D_\varepsilon - \log\frac{m^2}{\mu^2} \\
B_0\left(0,m_1^2,m_2^2\right) &= D_\varepsilon + 1 - \frac{1}{m_1^2-m_2^2}\left(m_1^2\log\frac{m_1^2}{\mu^2} - m_2^2\log\frac{m_2^2}{\mu^2}\right) \\
&= \frac{A_0\left(m_1^2\right)-A_0\left(m_2^2\right)}{m_1^2-m_2^2} \\
B_0\left(q^2,m^2,m^2\right) &= D_\varepsilon - \log\frac{m^2}{\mu^2} + H_1\left(\frac{q^2}{m^2}\right)
\end{aligned} \tag{66}$$

where $D_\varepsilon = 2/\varepsilon - \gamma + \log 4\pi$ and

$$H_1(a) = \begin{cases} 2 - 2\sqrt{4/a-1}\,\arctan\dfrac{1}{\sqrt{4/a-1}} & a<4 \\ 2 - \sqrt{1-4/a}\left(\log\dfrac{1+\sqrt{1-4/a}}{1-\sqrt{1-4/a}} - i\pi\right) & a>4 \end{cases} \tag{67}$$

The one-loop function defined in Ref. [19] is

$$\phi_i(q^2) = -\frac{4}{3}\frac{m_i^2}{q^2} + \frac{5}{18} + \frac{1}{3}\left(\frac{4m_i^2}{q^2}-1\right)\frac{2-H_1\left(q^2/m_i^2\right)}{2} \tag{68}$$

and for small q^2 can be expanded as

$$\phi_i(q^2) = -\frac{1}{6}\left[1 + \frac{q^2}{10m_i^2} + O\left(\frac{q^4}{m_i^4}\right)\right] \tag{69}$$

References

[1] S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D **2**, 1285 (1970).

[2] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).
M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[3] See e.g. G. Isidori, Int. J. Mod. Phys. A **17**, 3078 (2002) [hep-ph/0110255]; Annales Henri Poincare 4 (2003) S97 [hep-ph/0301159]; A. J. Buras, T. Ewerth, S. Jager and J. Rosiek, hep-ph/0408142.

[4] A. J. Buras, F. Schwab and S. Uhlig, hep-ph/0405132.

[5] G. Buchalla and A. J. Buras, Nucl. Phys. B **412** (1994) 106 [hep-ph/9308272].

[6] M. Misiak and J. Urban, Phys. Lett. B. **451** (1999) 161 [hep-ph/9901278].

[7] G. Buchalla and A. J. Buras, Nucl. Phys. B **548** (1999) 309 [hep-ph/9901288].

[8] G. Buchalla and G. Isidori, Phys. Lett. B. **440** (1998) 170 [hep-ph/9806501].

[9] A. F. Falk, A. Lewandowski and A. A. Petrov, Phys. Lett. B **505** (2001) 107 [hep-ph/0012099].

[10] D. Rein and L. M. Sehgal, Phys. Rev. D **39** (1989) 3325.

[11] J. S. Hagelin and L. S. Littenberg, Prog. Part. Nucl. Phys. **23** (1989) 1.

[12] M. Lu and M. B. Wise, Phys. Lett. B **324** (1994) 461 [hep-ph/9401204].

[13] C. Q. Geng, I. J. Hsu and Y. C. Lin, Phys. Lett. B **355** (1995) 569 [hep-ph/9506313]; Phys. Rev. D **50** (1994) 5744 [hep-ph/9406313].
C. Q. Geng, I. J. Hsu and C. W. Wang, Prog. Theor. Phys. **101** (1999) 937.

[14] S. Fajfer, Nuovo Cim. A **110** (1997) 397 [hep-ph/9602322].

[15] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).

[16] G. Ecker, Prog. Part. Nucl. Phys. **36**, 71 (1996) [hep-ph/9511412].
G. D'Ambrosio and G. Isidori, Int. J. Mod. Phys. A **13** (1998) 1 [hep-ph/9611284].

[17] J. Kambor, J. Missimer and D. Wyler, Nucl. Phys. B **346** (1990) 17.

[18] G. Ecker, J. Kambor and D. Wyler, Nucl. Phys. B **394** (1993) 101.

[19] G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B **291** (1987) 692.

[20] G. Ecker, A. Pich and E. de Rafael, Phys. Lett. B **189** (1987) 363.

[21] G. D'Ambrosio, G. Ecker, G. Isidori and J. Portolés, JHEP **08** (1998) 004 [hep-ph/9808289].

[22] G. Buchalla, G. D'Ambrosio and G. Isidori, Nucl. Phys. B **672** (2003) 387 [hep-ph/0308008].

[23] G. D'Ambrosio and G. Isidori, Phys. Lett. B **530**, 108 (2002) [hep-ph/0112135].